

Modular Categories associated to Unipotent Groups

Tanmay Deshpande

Abstract

Let G be a unipotent algebraic group over an algebraically closed field \mathbf{k} of characteristic $p > 0$ and let $l \neq p$ be another prime. Let e be a minimal idempotent in $\mathcal{D}_G(G)$, the braided monoidal category of G -equivariant (under conjugation action) $\overline{\mathbb{Q}}_l$ -complexes on G . Then we can associate to G and e a modular category $\mathcal{M}_{G,e}$. In this article, we prove that the modular categories that arise in this way from unipotent groups are precisely those in the class \mathfrak{C}_p^\pm .

1 Introduction

Let G be a unipotent group over an algebraically closed field \mathbf{k} of characteristic $p > 0$. We fix the field \mathbf{k} and all algebraic groups and schemes we consider will be assumed to be over \mathbf{k} . Let us also fix a prime number $l \neq p$. We have the $\overline{\mathbb{Q}}_l$ -linear triangulated monoidal category $\mathcal{D}(G)$ of $\overline{\mathbb{Q}}_l$ -complexes on G under convolution with compact support and the braided monoidal category $\mathcal{D}_G(G)$ of G -equivariant complexes under the conjugation action. The category $\mathcal{D}_G(G)$ is also equipped with a twist θ . (See [BD06, BD08] for details.) Let $e \in \mathcal{D}_G(G)$ be a minimal idempotent. Then we have the $\overline{\mathbb{Q}}_l$ -linear triangulated Hecke subcategory $e\mathcal{D}_G(G)$, which is a braided monoidal category with unit object e . In [BD08], Boyarchenko and Drinfeld define an \mathbb{L} -packet of character sheaves on G and a modular category corresponding to each such minimal idempotent e . We recall this below.

Theorem 1.1. *Let G be a unipotent group. Let $e \in \mathcal{D}_G(G)$ be a minimal idempotent. Then the following hold:*

- (i) *Let $\mathcal{M}_{G,e}^{perv} \subset e\mathcal{D}_G(G)$ denote the full subcategory consisting of objects of $e\mathcal{D}_G(G)$ whose underlying $\overline{\mathbb{Q}}_l$ -complex is a perverse sheaf on G . Then $\mathcal{M}_{G,e}^{perv}$ is a semisimple abelian category with finitely many simple objects. The (isomorphism classes of) simple objects of $\mathcal{M}_{G,e}^{perv}$ are said to form the \mathbb{L} -packet of character sheaves associated to the minimal idempotent e .*
- (ii) *There exists an integer $n_e \in \{0, 1, \dots, \dim(G)\}$ such that $e \in \mathcal{M}_{G,e}^{perv}[n_e]$. Moreover, the shifted subcategory $\mathcal{M}_{G,e} := \mathcal{M}_{G,e}^{perv}[n_e] \subset e\mathcal{D}_G(G)$ is closed under convolution and the twist θ in $\mathcal{D}_G(G)$ induces the structure of a modular category on $\mathcal{M}_{G,e}$.*

Remark 1.2. The number $d_e := \frac{\dim(G) - n_e}{2} \in \frac{1}{2}\mathbb{Z}$ is said to be the functional dimension of the minimal idempotent e .

Thus if we have a unipotent group G , and a minimal idempotent $e \in \mathcal{D}_G(G)$, we have the associated modular category $\mathcal{M}_{G,e}$. In this paper, we characterize all modular categories that arise in this way from unipotent groups, namely these modular categories are precisely those in the class \mathfrak{C}_p^\pm . Let us recall the definition below. (See also [DGNO2].)

Definition 1.3. A modular category \mathcal{C} is said to belong to \mathfrak{C}_p^\pm if the following conditions are satisfied:

- (i) The Frobenius-Perron dimension of \mathcal{C} equals p^{2k} for some $k \in \mathbb{Z}^+$.
- (ii) The categorical dimensions of all simple objects of \mathcal{C} are positive integers.
- (iii) The multiplicative central charge of \mathcal{C} is ± 1 .

If \mathcal{C} satisfies the conditions above and if the multiplicative central charge of \mathcal{C} is 1, we say $\mathcal{C} \in \mathfrak{C}_p^+$ and if the multiplicative central charge of \mathcal{C} is -1 , we say $\mathcal{C} \in \mathfrak{C}_p^-$.

It is proved in [DGNO2] that the class \mathfrak{C}_p^\pm may also be characterized as follows:

Theorem 1.4. [DGNO2] (i) A modular category $\mathcal{C} \in \mathfrak{C}_p^+$ if and only if it is equivalent as a modular category to the center (equipped with the positive spherical structure) of a pointed¹ fusion category whose group of isomorphism classes of simple objects is a finite p -group.

(ii) A modular category $\mathcal{C} \in \mathfrak{C}_p^-$ if and only if $\mathcal{C} \boxtimes \mathcal{M}_p^{anis} \in \mathfrak{C}_p^+$, where \mathcal{M}_p^{anis} is the modular category corresponding to the anisotropic metric group $(\mathbb{F}_{p^2}, \zeta^{N_{\mathbb{F}_{p^2}|\mathbb{F}_p}(\cdot)})$, where $\zeta \in \overline{\mathbb{Q}}_l^\times$ is a primitive p -th root and $N_{\mathbb{F}_{p^2}|\mathbb{F}_p} : \mathbb{F}_{p^2} \rightarrow \mathbb{F}_p$ is the norm.

The goal of this paper is to prove the following result conjectured by Drinfeld:

Theorem 1.5. (i) Let G be a unipotent group. Let $e \in \mathcal{D}_G(G)$ be a minimal idempotent with functional dimension $d_e \in \frac{1}{2}\mathbb{Z}$. Then if $d_e \in \mathbb{Z}$, $\mathcal{M}_{G,e} \in \mathfrak{C}_p^+$ and if $d_e \in \frac{1}{2} + \mathbb{Z}$, $\mathcal{M}_{G,e} \in \mathfrak{C}_p^-$.

(ii) If $\mathcal{C} \in \mathfrak{C}_p^\pm$, then there exists a connected unipotent group G with a minimal idempotent $e \in \mathcal{D}_G(G)$ such that $\mathcal{M}_{G,e} \cong \mathcal{C}$ as modular categories.

We prove Theorem 1.5(i) in §2. In [BD08], Boyarchenko and Drinfeld prove that every minimal idempotent $e \in \mathcal{D}_G(G)$ can be induced from a Heisenberg idempotent $e' \in \mathcal{D}_{G'}(G')$ satisfying a certain geometric Mackey condition, where G' is some subgroup of G . They also prove that $d_e = d_{e'} + \dim(G/G')$ and that there is an equivalence $\mathcal{M}_{G,e} \cong \mathcal{M}_{G',e'}$ of modular categories. In fact, it is shown in [B] that every minimal idempotent can be induced from a special Heisenberg idempotent. We recall this in §2.1 and hence reduce Theorem 1.5(i) to the case where e is a special Heisenberg idempotent. In §2.2 we complete the proof of Theorem 1.5(i) by proving it in the case of special Heisenberg idempotents.

In §3, we prove Theorem 1.5(ii). Let $\mathcal{C} \in \mathfrak{C}_p^\pm$. In §3.1, we show that it is enough to find a possibly disconnected unipotent group G with a Heisenberg idempotent $e \in \mathcal{D}_G(G)$ such that $\mathcal{C} \cong \mathcal{M}_{G,e}$. We show that if G is a (possibly disconnected) unipotent group with a Heisenberg idempotent e , then there exists a connected unipotent group U' and a minimal idempotent $f' \in \mathcal{D}_{U'}(U')$ such that $\mathcal{M}_{G,e} \cong \mathcal{M}_{U',f'}$. Finally, we complete the proof of Theorem 1.5(ii) for the case $\mathcal{C} \in \mathfrak{C}_p^+$ in §3.2 and the case $\mathcal{C} \in \mathfrak{C}_p^-$ in §3.3

Acknowledgments

I would like to thank V. Drinfeld for introducing me to this subject, for the many useful discussions and for suggesting corrections and improvements.

¹Pointedness means that all simple objects are invertible and hence the set of isomorphism classes of simple objects is a finite group, say Γ . All pointed fusion categories are of the form Vec_Γ^c , the category of finite dimensional Γ -graded vector spaces with usual tensor product, but with associativity constraints being given by a 3-cocycle ω .

2 Proof of Theorem 1.5(i)

In this section, we prove the first part of the main theorem. We begin by reducing to the case of Heisenberg idempotents and further to the case of special Heisenberg idempotents.

2.1 Reduction to special Heisenberg idempotents

Let us recall the notions of admissible pairs and Heisenberg idempotents defined in [BD06, BD08]. It will often be necessary to work with perfect schemes. For a scheme X over \mathbf{k} , we denote by X_{perf} its perfectization. For a perfect connected commutative unipotent group U over \mathbf{k} , we let U^* denote its Serre dual which parametrizes the multiplicative $\overline{\mathbb{Q}}_l$ -local systems on U .

Definition 2.1. Let G be a unipotent group. Let (N, \mathcal{L}) be a pair consisting of a connected subgroup $N \subset G$ and a multiplicative local system \mathcal{L} on N . We say that the pair (N, \mathcal{L}) is admissible if the following conditions hold:

- (i) Let G' denote the normalizer of the pair (N, \mathcal{L}) (see [BD08]) and let G'° denote its neutral connected component. Then G'°/N is commutative.
- (ii) The group morphism $\phi_{\mathcal{L}} : (G'^{\circ}/N)_{perf} \rightarrow (G'^{\circ}/N)_{perf}^*$ that is defined in this situation (see [BD08]) is an isogeny.
- (iii) (*Geometric Mackey condition*) For every $g \in G(\mathbf{k}) - G'(\mathbf{k})$, we have

$$\mathcal{L}|_{(N \cap N^g)^{\circ}} \not\cong \mathcal{L}^g|_{(N \cap N^g)^{\circ}},$$

where $N^g = g^{-1}Ng$ and \mathcal{L}^g is the multiplicative local system on N^g obtained from \mathcal{L} by transport of structure.

Remark 2.2. In the situation above, let $e' := \mathcal{L}[2 \dim N](\dim N) \in \mathcal{D}_{G'}(G')$. Then $e' \in \mathcal{D}_{G'}(G')$ is in fact a minimal idempotent (see [BD08]).

Definition 2.3. If (N, \mathcal{L}) is an admissible pair for G such that $G' = G$, we say that (N, \mathcal{L}) is a Heisenberg admissible pair² for G . In this case the minimal idempotent $e := \mathcal{L}[2 \dim N](\dim N) \in \mathcal{D}_G(G)$ is said to be a Heisenberg idempotent. Further, if we have $\dim(G/N) \leq 1$ we say that the idempotent is a special Heisenberg idempotent.

In [BD08], the induction (with compact support) functor $\text{ind}_{G'}^G : \mathcal{D}_{G'}(G') \rightarrow \mathcal{D}_G(G)$ is defined for closed subgroups $G' \subset G$. In the following theorem, Boyarchenko and Drinfeld prove that every minimal idempotent in $\mathcal{D}_G(G)$ comes from an admissible pair and in particular from a Heisenberg idempotent on a subgroup by induction.

Theorem 2.4. [BD08] (i) Let (N, \mathcal{L}) be an admissible pair for a unipotent group G and let $e' \in \mathcal{D}_{G'}(G')$ be the corresponding Heisenberg idempotent on G' as defined above. Then $e := \text{ind}_{G'}^G e' \in \mathcal{D}_G(G)$ is a minimal idempotent. The functional dimensions of e and e' are related as follows:

$$d_e = d'_{e'} + \dim(G/G').$$

²Note that in this situation, condition (iii) of the definition is vacuous.

(ii) In the situation of (i), the induction functor induces an equivalence of modular categories $\mathcal{M}_{G',e'} \cong \mathcal{M}_{G,e}$.

(iii) Every minimal idempotent $e \in \mathcal{D}_G(G)$ comes from an admissible pair by the procedure described in (i). Hence every minimal idempotent $e \in \mathcal{D}_G(G)$ comes from induction from a Heisenberg idempotent e' on some subgroup G' . In fact, we may choose e', G' such that e' is a special Heisenberg idempotent on G' . (See [B].)

Hence we see that every minimal idempotent $e \in \mathcal{D}_G(G)$ comes from induction from a special Heisenberg idempotent e' on some subgroup G' satisfying the Geometric Mackey condition. Moreover, we have an equivalence $\mathcal{M}_{G',e'} \cong \mathcal{M}_{G,e}$ of modular categories and the functional dimensions of e and e' differ by an integer. Thus, it is enough to prove Theorem 1.5(i) for special Heisenberg idempotents.

2.2 Proof in the case of special Heisenberg idempotents

Let G be a unipotent group. Let (N, \mathcal{L}) be a Heisenberg admissible pair for G and let $e = \mathcal{L}[2 \dim N](\dim N) \in \mathcal{D}_G(G)$ be the corresponding Heisenberg idempotent. Let $H = G^\circ$ and let $\Gamma = G/H$. Since G is unipotent, Γ is a p -group. We see that in this case $n_e = \dim(N)$ and hence $d_e = \frac{\dim(H/N)}{2}$. Note that we have the skew-symmetric isogeny $\phi_{\mathcal{L}} : (H/N)_{\text{perf}} \rightarrow (H/N)_{\text{perf}}^*$. Let $(K_{\mathcal{L}}, \theta)$ denote the corresponding metric group, where $K_{\mathcal{L}} = \ker \phi_{\mathcal{L}}$. We also have an action of Γ on this metric group. In [De], it is shown that the modular category $\mathcal{M}_{H,e}$ is equivalent to the modular category defined by the metric group $(K_{\mathcal{L}}, \theta)$. Let $\widetilde{\mathcal{M}}_{G,e}$ denote the full subcategory of $e\mathcal{D}_H(G)$ consisting of perverse sheaves shifted by $\dim N$. In [De], it is proved that $\widetilde{\mathcal{M}}_{G,e}$ is closed under convolution and that we have a braided Γ -crossed structure on $\widetilde{\mathcal{M}}_{G,e}$ with trivial component $\mathcal{M}_{H,e}$ and that $\mathcal{M}_{G,e} \cong \widetilde{\mathcal{M}}_{G,e}^\Gamma$.

Remark 2.5. We note that whenever we have a minimal idempotent e on a unipotent group G , we have the modular category $\mathcal{M}_{G,e}$. Moreover, if e is also a Heisenberg idempotent as described above, then we have also have a braided Γ -crossed category $\widetilde{\mathcal{M}}_{G,e}$ whose identity component is the pointed modular category $\mathcal{M}_{H,e}$ and whose equivariantization $(\widetilde{\mathcal{M}}_{G,e})^\Gamma \cong \mathcal{M}_{G,e}$.

We now prove the following proposition for all Heisenberg idempotents, and later we restrict to the class of special Heisenberg idempotents.

Proposition 2.6. (i) The Gauss sums $\tau^\pm(\mathcal{M}_{G,e})$ are equal to $(-1)^{2d_e} p^k \cdot |\Gamma|$ for some integer $k \geq 0$. Hence the multiplicative central charge of $\mathcal{M}_{G,e}$ equals $(-1)^{2d_e}$.

(ii) The Frobenius-Perron dimension of $\mathcal{M}_{G,e}$ equals $|\Gamma|^2 \cdot |K_{\mathcal{L}}|$.

Proof. Since $\mathcal{M}_{H,e}$ is the modular category corresponding to the metric group $(K_{\mathcal{L}}, \theta)$, by [Da, Prop. 5] we have $\tau^\pm(\mathcal{M}_{H,e}) = \tau^\pm(K_{\mathcal{L}}, \theta) = (-1)^{2d_e} p^k$ for some integer $k \geq 0$. Suppose $(M, \phi) \in \mathcal{M}_{H,e}^\Gamma \subset \mathcal{M}_{G,e}$, where $M \in \mathcal{M}_{H,e}$ and ϕ is the equivariant structure. The twist $\theta_{(M, \phi)}$ in $\mathcal{M}_{G,e}$ is the same as the twist θ_M in $\mathcal{M}_{H,e}$ as an automorphism of M . Let \mathcal{E} be the full subcategory of $\mathcal{M}_{H,e}^\Gamma$ consisting of objects (M, ϕ) such that M is isomorphic to a direct sum of copies of the unit object e . Hence it follows that the twist when restricted to \mathcal{E} is trivial, or in other words, \mathcal{E} is an isotropic subcategory of $\mathcal{M}_{G,e}$. Moreover, it is clear that $\mathcal{E} \cong \text{Rep}(\Gamma)$. The category $\mathcal{M}_{H,e}$ is the corresponding fiber category. Hence by [DGNO, Thm. 6.16], $\tau^\pm(\mathcal{M}_{G,e}) = \tau^\pm(\mathcal{M}_{H,e}) \cdot |\Gamma|$.

Statement (i) now follows, since $\tau^\pm(\mathcal{M}_{H,e}) = (-1)^{2d_e} p^k$. To prove (ii), note that $\text{FPdim}(\widetilde{\mathcal{M}}_{G,e}) = |\Gamma| \cdot \text{FPdim}(\mathcal{M}_{H,e}) = |\Gamma| \cdot |K_{\mathcal{L}}|$ by [DGNO, Prop. 2.21] and that $\text{FPdim}(\widetilde{\mathcal{M}}_{G,e}^\Gamma) = |\Gamma| \cdot \text{FPdim}(\mathcal{M}_e)$ by [DGNO, 4.26]. \square

Remark 2.7. Since $(K_{\mathcal{L}}, \theta)$ is a metric group coming from a skew-symmetric biextension, it follows from [Da] that $|K_{\mathcal{L}}|$ is an even power of p . Moreover, Γ is a p -group. Hence we see that $\text{FPdim}(\mathcal{M}_{G,e}) = p^{2k}$ for some $k \in \mathbb{Z}^+$. We also see that the multiplicative central charge of $\mathcal{M}_{G,e}$ is 1 if $d_e \in \mathbb{Z}$ and -1 if $d_e \in \frac{1}{2} + \mathbb{Z}$. Hence to prove Theorem 1.5(i), it only remains to prove that the categorical dimensions of all simple objects of $\mathcal{M}_{G,e}$ are positive integers. We prove this in Proposition 2.9 below.

Let us now assume that the Heisenberg admissible pair (N, \mathcal{L}) is also special, i.e. $\dim(H/N) \leq 1$. Hence $H/N = 0$ or \mathbb{G}_a . Hence the induced action of the p -group Γ on H/N is trivial. This means that the commutator map from $H \times G$ in fact maps to N . We will now use some of the tools developed in [De]. For $g \in G$ we have the commutator map $c_g : H \rightarrow N$ defined $h \mapsto hgh^{-1}g^{-1}$. Also, we recall that in this situation, the objects of $e\mathcal{D}_H(G)$ are supported on $\widetilde{K} := \{g \in G | c_g^* \mathcal{L} \cong \overline{\mathbb{Q}}_l\}$. We can readily modify the proof of Proposition 4.6 from [De] and prove the following:

Proposition 2.8. *Let $k \in \widetilde{K} \subset G$. Let e^k denote the right translate of e by k . Then $e^k \in \widetilde{\mathcal{M}}_{G,e} \subset e\mathcal{D}_H(G)$. The isomorphism class of e^k only depends on the coset $Nk \in \widetilde{K}/N$ and the $e^k, k \in \widetilde{K}$ are all the simple objects of the braided Γ -crossed category $\widetilde{\mathcal{M}}_{G,e}$. Hence $\widetilde{\mathcal{M}}_{G,e}$ is pointed with the group of isomorphism classes of simple objects being \widetilde{K}/N .*

We now complete the proof of Theorem 1.5(i) by proving the following:

Proposition 2.9. *The categorical dimensions of all the simple objects of $\widetilde{\mathcal{M}}_{G,e}$ are 1. Hence the categorical dimensions of all simple objects of $\mathcal{M}_{G,e} \cong \widetilde{\mathcal{M}}_{G,e}^\Gamma$ are positive integers.*

Proof. We must prove that $\dim(e^k) = 1$ for all $k \in \widetilde{K}$. Note that e^k is supported on Nk and that its dual $(e^k)^\vee = \mathbb{D}\iota^*(e^k)[2\dim N](\dim N)$ (see [De]) is supported on $k^{-1}N$. Here \mathbb{D} denotes Verdier duality and $\iota : G \rightarrow G$ is the inversion map. Let us identify Nk and $k^{-1}N$ with N by the evident maps. Under this identification $\iota : Nk \rightarrow k^{-1}N$ gets identified with $\iota : N \rightarrow N$, $\mu : Nk \times k^{-1}N \rightarrow N$ gets identified with the multiplication for N (hence this identification is compatible with convolutions), e^k gets identified with e and Verdier duality is also compatible. Hence we see that $\dim(e^k) = \dim(e) = 1$. Hence categorical dimensions of all simple objects of $\widetilde{\mathcal{M}}_{G,e}$ are 1. Moreover, the categorical dimensions of all simple objects of $\mathcal{M}_{G,e} \cong \widetilde{\mathcal{M}}_{G,e}^\Gamma$ are positive integral multiples of those of $\widetilde{\mathcal{M}}_{G,e}$ (see [DGNO, 4.26]). Hence the categorical dimensions of all simple objects of $\mathcal{M}_{G,e}$ must also be positive integers. \square

3 Proof of Theorem 1.5(ii)

Let $\mathcal{C} \in \mathfrak{C}_p^\pm$. In order to prove Conjecture 1.5(ii), we first show that it is enough to find a possibly disconnected unipotent group G with a Heisenberg idempotent e such that $\mathcal{C} \cong \mathcal{M}_{G,e}$.

3.1 Passing from a disconnected group to a connected group

Let G be a possibly disconnected unipotent group with a Heisenberg admissible pair (N, \mathcal{L}) . As before, let $H = G^0$. Let $e \in \mathcal{D}_G(G)$ denote the corresponding Heisenberg idempotent. We will prove that there exists a connected unipotent group U' and a minimal idempotent $f' \in \mathcal{D}_{U'}(U')$ such that the modular categories $\mathcal{M}_{G,e}$ and $\mathcal{M}_{U',f'}$ are equivalent.

Proposition 3.1. *Let V be a unipotent group with an action of G by automorphisms. Let \mathcal{K} be a G -equivariant multiplicative local system on V . Then we have the following:*

- (a) $(V \rtimes N, \mathcal{K} \boxtimes \mathcal{L})$ is a Heisenberg admissible pair for $V \rtimes G$.
- (b) We have an equivalence $e\mathcal{D}_G(G) \cong (f \boxtimes e)\mathcal{D}_{(V \rtimes G)}(V \rtimes G)$, where $f = \mathcal{K}[2 \dim V](\dim V)$, given by $M \mapsto f \boxtimes M$. Here $f \boxtimes e$ is the Heisenberg idempotent corresponding to the admissible pair above.

Proof. Since \mathcal{K} is G -equivariant multiplicative local system (and hence N -equivariant) it is easy to check that $\mathcal{K} \boxtimes \mathcal{L}$ is a $V \rtimes G$ -equivariant multiplicative local system on $V \rtimes N$. We also check that the corresponding morphism $((V \rtimes H)/(V \rtimes N))_{perf} \rightarrow ((V \rtimes H)/(V \rtimes N))_{perf}^*$ can be identified with the map $(H/N)_{perf} \rightarrow (H/N)_{perf}^*$ corresponding to the admissible pair (N, \mathcal{L}) . For part (b), we use the fact that f is G -equivariant. \square

Let us now proceed to construct the connected unipotent group U' . First, we embed G into a connected unipotent group U . Recall the well known fact that given an algebraic group along with a closed subgroup, we can find a representation of the group on a vector space and a line in the vector space such that the subgroup can be characterised as the one mapping the line to itself. Hence we can find a vector space V with a U -action and a multiplicative local system \mathcal{K} on V (in other words $\mathcal{K} \in V^*$, which can be thought of as the vector space we start with) such that G is precisely the stabilizer of $\mathcal{K} \in V^*$ under the U -action. Now let $U' = V \rtimes U$. Let $N' = V \rtimes N$ and $\mathcal{L}' = \mathcal{K} \boxtimes \mathcal{L}$. We will now prove that the pair (N', \mathcal{L}') is admissible for U' , and that its normalizer is $G' = V \rtimes G$.

Proposition 3.2. (a) For $u \in U' - G'$ we have

$$\mathcal{L}'|_{(N' \cap {}^u N')^0} \not\cong {}^u \mathcal{L}'|_{(N' \cap {}^u N')^0}.$$

- (b) G' is the normalizer of the pair (N', \mathcal{L}') and it is an admissible pair for U' .
- (c) Let $f' \in \mathcal{D}_{U'}(U')$ be the corresponding minimal idempotent. Then we have an equivalence $e\mathcal{D}_G(G) \cong f'\mathcal{D}_{U'}(U')$.

Proof. (a) Note that since V is a connected normal subgroup of U' , we have $V \subset (N' \cap {}^u N')^0$. We have $\mathcal{L}'|_V \cong \mathcal{K}$ and that ${}^u \mathcal{L}'|_V \cong {}^u \mathcal{K}$. We have chosen \mathcal{K} such that the stabilizer of $\mathcal{K} \in V^*$ under the action of U is precisely G . Hence we have that ${}^u \mathcal{K} \not\cong \mathcal{K}$ for each $u \in U' - G'$. Hence (a) follows. (b) It is clear that G' is contained in the normalizer of (N', \mathcal{L}') . But from (a) we see that it is precisely the normalizer. This combined with (a) and Prop. 3.1 implies that (N', \mathcal{L}') is an admissible pair for U' . (c) Note that we have $f' = \text{ind}_{G'}^{U'}(f \boxtimes e)$, where f is as in Prop. 3.1. Hence (c) follows from Prop. 3.1(b) and Theorem 2.4(ii). \square

3.2 The case $\mathcal{C} \in \mathfrak{C}_p^+$

Let $\mathcal{C} \in \mathfrak{C}_p^+$. By Theorem 1.4, we have a realization $\mathcal{C} \cong \mathcal{Z}(\text{Vec}_\Gamma^\omega)$ equipped with positive spherical structure for some finite p -group Γ and a 3-cocycle $\omega : \Gamma^3 \rightarrow \overline{\mathbb{Q}}_l^\times$ for the trivial action of Γ on $\overline{\mathbb{Q}}_l^\times$. Since Γ is a finite p -group, we may assume that ω takes values in a finite subgroup A_n of $\overline{\mathbb{Q}}_l^\times$ of p^n -th roots of unity for some n .

We will now construct a unipotent group G along with a Heisenberg idempotent e on it such that (see Remark 2.5) $\widetilde{\mathcal{M}}_{G,e} = \text{Vec}_\Gamma^\omega$. Hence as required we will have $\mathcal{M}_{G,e} \cong (\text{Vec}_\Gamma^\omega)^\Gamma \cong \mathcal{Z}(\text{Vec}_\Gamma^\omega) \cong \mathcal{C}$. Let W_n denote the ring of Witt vectors of length n . Consider the group ring $\widetilde{H} := W_n(\Gamma) = \text{Maps}(\Gamma, W_n)$. This is a commutative unipotent group (isomorphic to $W_n^{|\Gamma|}$) with an action of Γ by left translations. We embed $A_n \hookrightarrow W_n \xrightarrow{\Delta} \widetilde{H}$ via the diagonal. Note that under this embedding Γ acts trivially on $A_n \subset \widetilde{H}$. Let $H := \widetilde{H}/A_n$ as a Γ -module. Hence we have an exact sequence

$$0 \rightarrow A_n \rightarrow \widetilde{H} \rightarrow H \rightarrow 0 \quad (1)$$

of Γ -modules. It is a Γ -equivariant central extension of H by $A_n \subset \overline{\mathbb{Q}}_l^\times$. Let \mathcal{L} denote the corresponding Γ -equivariant multiplicative local system on H . Note that since \widetilde{H} is the group algebra, $H^i(\Gamma, \widetilde{H}) = 0$ for $i > 0$ by Shapiro's lemma. Hence from the long exact sequence of cohomology groups associated to the above short exact sequence, we get an isomorphism $H^2(\Gamma, H) \rightarrow H^3(\Gamma, A_n)$. Hence there exists a 2-cocycle $f : \Gamma^2 \rightarrow H$ that maps to the class of ω and we get a corresponding extension

$$0 \rightarrow H \rightarrow G \rightarrow \Gamma \rightarrow 0. \quad (2)$$

Note that (H, \mathcal{L}) is a Heisenberg admissible pair. Let e be the corresponding idempotent. Then we see that $\widetilde{\mathcal{M}}_{G,e} \cong \text{Vec}_\Gamma^\omega$ and hence $\mathcal{M}_{G,e} \cong \mathcal{Z}(\text{Vec}_\Gamma^\omega)$.

3.3 The case $\mathcal{C} \in \mathfrak{C}_p^-$

We begin with an alternative description of the modular categories in \mathfrak{C}_p^- .

3.3.1 An alternative description of \mathfrak{C}_p^-

We will use the following result from [DGNO2] that gives another characterization of the class \mathfrak{C}_p^- .

Theorem 3.3. [DGNO2] *A modular category $\mathcal{C} \in \mathfrak{C}_p^-$ if and only if it can be realized as $\widetilde{\mathcal{M}}^\Gamma$ where Γ is a p -group and $\widetilde{\mathcal{M}}$ is a faithfully graded braided Γ -crossed category with trivial component $\mathcal{M}_p^{\text{anis}}$ such that the induced action of Γ on the anisotropic metric group $(\mathbb{F}_{p^2}, \zeta^N)$ is trivial.*

Braided Γ -crossed categories as in the theorem above are classified (see [ENO]) by pairs (f, α) where $f \in H^2(\Gamma, \mathbb{F}_{p^2})$ such that a certain obstruction in $H^4(\Gamma, \overline{\mathbb{Q}}_l^\times)$ vanishes, and α lies in a certain torsor over $H^3(\Gamma, \overline{\mathbb{Q}}_l^\times)$.

Let $\widetilde{\mathcal{M}}$ be such a braided Γ -crossed category corresponding to an $f \in H^2(\Gamma, \mathbb{F}_{p^2})$ (such that the corresponding obstruction vanishes). Then we wish to construct a unipotent group G with a Heisenberg idempotent e such that $\widetilde{\mathcal{M}}_{G,e} \cong \widetilde{\mathcal{M}}$.

Let us now show that if we can realize one such braided Γ -crossed category corresponding to f in this way, then we can realize all.

Proposition 3.4. *Let $\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}'$ be braided Γ -crossed categories corresponding to $f \in H^2(\Gamma, \mathbb{F}_{p^2})$. Suppose there exists a unipotent group G with a Heisenberg idempotent e such that $\widetilde{\mathcal{M}}_{G,e} \cong \widetilde{\mathcal{M}}$. Then $\widetilde{\mathcal{M}}'$ can also be realized from some Heisenberg idempotent on some unipotent group.*

Proof. Indeed $\widetilde{\mathcal{M}}'$ differs from $\widetilde{\mathcal{M}}$ in the associativity constraint by an element $\omega \in H^3(\Gamma, \overline{\mathbb{Q}}_l^\times)$. By assumption, we have an extension $0 \rightarrow H \rightarrow G \rightarrow \Gamma \rightarrow 0$ with a Heisenberg admissible pair (N, \mathcal{L}) for G that realizes $\widetilde{\mathcal{M}}$. By §3.2, we also have a commutative unipotent group V equipped with a Γ -action and a Γ -equivariant multiplicative local system \mathcal{K} on V and an extension $0 \rightarrow V \rightarrow W \rightarrow \Gamma \rightarrow 0$ corresponding to the 3-cocycle ω . Let us consider the product of the group extensions:

$$0 \rightarrow H \times V \rightarrow U \rightarrow \Gamma \rightarrow 0.$$

Now $(N \times V, \mathcal{L} \boxtimes \mathcal{K})$ is an admissible pair for U and the corresponding braided Γ -crossed category is $\widetilde{\mathcal{M}}'$. \square

Hence in order to complete the proof of Theorem 1.5(ii), given a 2-cocycle $f \in H^2(\Gamma, \mathbb{F}_{p^2})$, or equivalently a central extension $0 \rightarrow \mathbb{F}_{p^2} \rightarrow K_\Gamma \rightarrow \Gamma \rightarrow 0$ (such that the corresponding obstruction vanishes) it is enough to construct a unipotent group U with $\pi_0(U) = \Gamma$ with a Heisenberg idempotent such that the corresponding braided Γ -crossed category is one corresponding to the given central extension.

3.3.2 The central extension and obstruction in two step nilpotent case

Let H be a two step nilpotent unipotent group with connected center N and H/N commutative. Let Γ be a finite p -group with an outer action on H given by a group homomorphism $\Gamma \rightarrow \text{Out}(H)$. This induces an honest action of Γ on the center N as well as on $H/N = \text{Inn}(H)$ since we have assumed that $\text{Inn}(H)$ is commutative. Let us assume that the induced action of Γ on H/N is trivial. Let \mathcal{L} be a Γ -equivariant multiplicative local system on N , such that (N, \mathcal{L}) is a Heisenberg admissible pair on H . Hence the induced (Γ -equivariant) skew-symmetric biextension map $\phi_{\mathcal{L}} : (H/N)_{\text{perf}} \rightarrow (H/N)_{\text{perf}}^*$ is an isogeny with kernel $K_{\mathcal{L}} = K/N \subset H/N$ and we have the corresponding metric θ on $K_{\mathcal{L}}$ and the Heisenberg idempotent e on H . We can also define a braided action of Γ on $e\mathcal{D}_H(H)$ and hence on $\mathcal{M}_{H,e}$.

Let us pullback the exact sequence $0 \rightarrow \text{Inn}(H) \rightarrow \text{Aut}(H) \rightarrow \text{Out}(H) \rightarrow 0$ along the outer action map and obtain the central (since induced action of Γ on $\text{Inn}(H)$ is trivial) extension

$$0 \rightarrow \text{Inn}(H) \rightarrow \text{Aut}_\Gamma(H) \rightarrow \Gamma \rightarrow 0.$$

For $\gamma \in \Gamma$, let $\text{Aut}_\gamma(H) \subset \text{Aut}(H)$ be the $\text{Inn}(H)$ -coset that describes the outer action of γ on H . For $\tilde{\gamma} \in \text{Aut}_\gamma(H)$, let $c_{\tilde{\gamma}} : H \rightarrow N$ be the map $c_{\tilde{\gamma}}(h) = h\tilde{\gamma}(h^{-1}) \in N$. ($c_{\tilde{\gamma}}(h) \in N$ since Γ acts trivially on H/N .) We can check that $c_{\tilde{\gamma}}$ is a group homomorphism. Hence $c_{\tilde{\gamma}}^*\mathcal{L}$ is a multiplicative local system on H that is trivial when restricted to N since \mathcal{L} is Γ -equivariant. Hence we have a map $\text{Aut}_\gamma(H) \rightarrow (H/N)^*$ given by $\tilde{\gamma} \mapsto c_{\tilde{\gamma}}^*\mathcal{L}$. Also note that we have

$$c_{\tilde{\gamma}_1 \tilde{\gamma}_2}(h) = c_{\tilde{\gamma}_1}(h) \gamma_1(c_{\tilde{\gamma}_2}(h)),$$

hence we get a homomorphism of groups $\phi_\Gamma : \text{Aut}_\Gamma(H) \rightarrow (H/N)^*$. Note that $\text{Aut}_\gamma(H)$ is a torsor over $\text{Inn}(H) = H/N$. Let $K_\gamma \subset \text{Aut}_\gamma(H)$ be the set of elements that map to the identity in $(H/N)^*$

under the above map. Let K_Γ be the disjoint union of all the K_γ . We have $K_\Gamma = \ker(\phi_\Gamma)$ and we have a central extension

$$0 \rightarrow K_1 = K_\mathcal{L} \rightarrow K_\Gamma \rightarrow \Gamma \rightarrow 0.$$

Thus from the outer action of Γ on H and the Γ -equivariant multiplicative local system \mathcal{L} on the center N , we can get a two cocycle $f : \Gamma \times \Gamma \rightarrow K_\mathcal{L} \subset H/N = \text{Inn}(H)$ by choosing a $\tilde{\gamma} \in K_\gamma \subset \text{Aut}_\gamma(H) \subset \text{Aut}(H)$ for each $\gamma \in \Gamma$ such that

$$\tilde{\gamma}_1 \tilde{\gamma}_2 = f(\gamma_1, \gamma_2) \widetilde{\gamma_1 \gamma_2}.$$

Note that $c_{\tilde{\gamma}}^* \mathcal{L} \cong \overline{\mathbb{Q}}_l$ by definition of K_γ and we have the following relation between the different $c_{\tilde{\gamma}}$:

$$c_{\tilde{\gamma}_1}(h)^{\gamma_1} c_{\tilde{\gamma}_2}(h) = [h, f(\gamma_1, \gamma_2)] c_{\widetilde{\gamma_1 \gamma_2}}(h).$$

Let $F : \Gamma \times \Gamma \rightarrow K \subset H$ be a lift of f to $K \subset H$. Hence we get a 3-cocycle $\omega : \Gamma^3 \rightarrow N$ such that we have

$$\tilde{g}(F(h, k))F(g, hk) = \omega(g, h, k)F(g, h)F(gh, k), \text{ for } g, h, k \in \Gamma.$$

Thus starting from an outer action of Γ on H we get an $\omega \in H^3(\Gamma, N)$. This is the obstruction to the existence of an extension of Γ by H corresponding to given outer action.

If we also have a Γ -equivariant Heisenberg admissible pair (N, \mathcal{L}) as above, then we get a braided action of Γ on $\mathcal{M}_{H,e}$ and an extension $0 \rightarrow K_\mathcal{L} \rightarrow K_\Gamma \rightarrow \Gamma \rightarrow 0$. This data gives rise to an obstruction $\beta \in H^4(\Gamma, \overline{\mathbb{Q}}_l^*)$ to the existence of a braided Γ -crossed category with trivial component $\mathcal{M}_{H,e}$. On the other hand we have the boundary map $\delta : H^3(\Gamma, N) \rightarrow H^4(\Gamma, \overline{\mathbb{Q}}_l^*)$ coming from the Γ -equivariant multiplicative local system \mathcal{L} on N (which can be thought of as a short exact sequence of Γ -modules $0 \rightarrow \overline{\mathbb{Q}}_l^* \rightarrow \tilde{N} \rightarrow N \rightarrow 0$). Next we show that $\delta\omega = \beta$.

3.3.3 The cohomological obstructions

In the previous section, we described a 3-cocycle $\omega \in H^3(\Gamma, N)$ which is the obstruction to the existence of an extension of Γ by H corresponding to the given outer action. We have a Γ -equivariant multiplicative local system \mathcal{L} on N . We note that the outer action respected the Heisenberg admissible pair (N, \mathcal{L}) on H and that the induced action of Γ on H/N (and hence on $K_\mathcal{L} \subset H/N$) was trivial. Using this we obtained a braided action of Γ on $\mathcal{M}_{H,e}$ and a central extension

$$0 \rightarrow K_\mathcal{L} \rightarrow K_\Gamma \rightarrow \Gamma \rightarrow 0$$

say corresponding to $f \in H^2(\Gamma, K_\mathcal{L})$. We thus get a $\beta \in H^4(\Gamma, \overline{\mathbb{Q}}_l^*)$ which is the obstruction to the existence of the corresponding braided Γ -crossed category.

Let us describe β explicitly. We continue to use all the notations from the previous section. We have the braided action of Γ on $\mathcal{M}_{H,e}$. Then we have the Pontryagin-Whitehead quadratic function $PW : H^2(\Gamma, K_\mathcal{L}) \rightarrow H^4(\Gamma, \overline{\mathbb{Q}}_l^*)$ as described in [ENO, §8.7] and $\beta = PW(f)$. By [De, §4.3], the simple objects of $\mathcal{M}_{H,e}$ can be described as the translates e^k of e by $k \in K \subset H$ and moreover, for $k_1, k_2 \in K$ we can identify $e^{k_1} * e^{k_2}$ with $e^{k_1 k_2}$. The isomorphism class of e^k only depends on the coset Nk . For $n \in N$ and any $k \in K$, a choice of trivialization of the stalk \mathcal{L}_n gives us an isomorphism $e^k \rightarrow e^{nk}$. For $a, b, c \in \Gamma$, let us choose a trivialization of the stalk $\mathcal{L}_{\omega(a,b,c)}$. This gives us an isomorphism $\zeta_{a,b,c} : e \rightarrow e^{\omega(a,b,c)}$. Consider the map L from $\Gamma \times \Gamma$ to simple objects of $\mathcal{M}_{H,e}$ given by $L_{a,b} = e^{F(a,b)}$, where $F : \Gamma \times \Gamma \rightarrow K$ is the lift of the 2-cocycle f as in the previous

section. Note that for $a \in \Gamma$, we have chosen lifts $\tilde{a} \in K_a \subset \text{Aut}_a(H)$. We can define the braided action of Γ on $\mathcal{M}_{H,e}$ by $a(e^k) = e^{\tilde{a}(k)}$.

Let us now compute $PW(f)$. For this, note that we have chosen isomorphisms (up to a small abuse of notation)

$$\zeta_{a,b,c} : L_{a,b} * L_{ab,c} = e^{F(a,b)F(ab,c)} \rightarrow a(L_{b,c}) * L_{a,bc} = e^{\tilde{a}F(b,c)F(a,bc)} = e^{\omega(a,b,c)F(a,b)F(ab,c)}.$$

Hence according to [ENO, §8.7], for $a, b, c, d \in \Gamma$, $PW(f)(a, b, c, d)$ is given by the automorphism of $L_{a,b} * L_{ab,c} * L_{abc,d} = e^{F(a,b)F(ab,c)F(abc,d)}$ given by the composition

$$\begin{aligned} L_{a,b} * L_{ab,c} * L_{abc,d} &\rightarrow a(L_{b,c}) * L_{a,bc} * L_{abc,d} \rightarrow a(L_{b,c}) * a(L_{bc,d}) * L_{a,bcd} \\ &\rightarrow a(b(L_{c,d})) * a(L_{b,cd}) * L_{a,bcd} \rightarrow ab(L_{c,d}) * L_{a,b} * L_{ab,cd} \\ &\rightarrow L_{a,b} * ab(L_{c,d}) * L_{ab,cd} \rightarrow L_{a,b} * L_{ab,c} * L_{abc,d}, \end{aligned}$$

where the all isomorphisms are given by the various ζ and their inverses, the braiding, and the natural isomorphism between the braided monoidal functors $a \circ b$ and ab .

On the other hand we have the Γ -equivariant multiplicative local system \mathcal{L} on N . This gives us the boundary map $\delta : H^3(\Gamma, N) \rightarrow H^4(\Gamma, \mathbb{Q}_l^*)$. Let us describe $\delta\omega$ more explicitly. Note that we have $\omega(ab, c, d)^{-1} \cdot \omega(a, b, cd)^{-1} \cdot \tilde{a}\omega(b, c, d) \cdot \omega(a, bc, d) \cdot \omega(a, b, c) = 1$. Hence using our chosen trivializations of $\mathcal{L}_{\omega(\cdot, \cdot, \cdot)}$, we get an automorphism of \mathcal{L}_1 . This automorphism is $\delta\omega(a, b, c, d)$. Using this, we see that $\delta\omega = PW(f) = \beta$.

3.3.4 Constructing an outer action from a central extension

Let Γ be a finite p -group and let

$$0 \rightarrow \mathbb{F}_{p^2} \rightarrow K_\Gamma \rightarrow \Gamma \rightarrow 0$$

be a central extension. We will now construct a two step nilpotent group H as above with an action of K_Γ such that $\mathbb{F}_{p^2} \subset H/N = \text{Inn}(H)$ acts by its natural conjugation action on H (this induces an outer action of Γ on H) and a Γ -equivariant Heisenberg admissible pair (N, \mathcal{L}) as above, such that the corresponding metric group is the anisotropic metric group $(\mathbb{F}_{p^2}, \theta)$ and such that the corresponding extension of Γ by $K_\mathcal{L} = \mathbb{F}_{p^2}$ obtained as before coincides with K_Γ .

Let $\tilde{N} = \text{Maps}(\Gamma, \mathbb{G}_a)$, or the group algebra of Γ over the ring \mathbb{G}_a considered as a commutative unipotent group with a Γ -action by left multiplication. Note that we have $H^i(\Gamma, \tilde{N}) = 0$ for $i > 0$ by Shapiro's lemma. Note that we have the embedding $\mathbb{F}_p \subset \mathbb{G}_a \xrightarrow{\Delta} \tilde{N}$ of constant maps which is a Γ -equivariant map with Γ acting trivially on \mathbb{G}_a . Let $N := \tilde{N}/\mathbb{F}_p$. Hence we get a Γ -equivariant central extension

$$0 \rightarrow \mathbb{F}_p \rightarrow \tilde{N} \rightarrow N \rightarrow 0,$$

or in other words, a Γ -equivariant multiplicative local system, \mathcal{L} on N (after choosing a nontrivial character of \mathbb{F}_p).

Note that we have $\mathbb{G}_a \cong \mathbb{G}_a/\mathbb{F}_p \subset \tilde{N}/\mathbb{F}_p = N$. Thus we have an embedding $\mathbb{G}_a \subset N$ and Γ acts trivially on \mathbb{G}_a under this embedding and $\mathcal{L}|_{\mathbb{G}_a}$ is just the Artin-Schreier local system. Let $B : \mathbb{G}_a \times \mathbb{G}_a \rightarrow \mathbb{G}_a \subset N$ be the skew-symmetric biadditive map given by $B(x, y) = xy^p - x^py \in \mathbb{G}_a \subset N$. Hence the local system \mathcal{L} on N induces a skew-symmetric biextension $\mathbb{G}_a \rightarrow (\mathbb{G}_a)^*$ which is an isogeny with kernel \mathbb{F}_{p^2} and the induced metric is the anisotropic metric.

Now let H be the fake Heisenberg group which is $\mathbb{G}_a \times N$ as a variety and $(x, n) \cdot (y, m) = (x + y, n + m + xy^p)$. (Here $xy^p \in \mathbb{G}_a \subset N$ as described above.) The center of H is N and $\text{Inn}(H) = H/N = \mathbb{G}_a$ and the inner action of \mathbb{G}_a on H is given by

$$\mathbb{G}_a \ni x : (y, m) \mapsto (y, m + B(x, y)).$$

Note that we have $(x, n)^{-1} = (-x, x^{p+1} - n)$ and that the commutator map $[\cdot, \cdot] : H \times H \rightarrow N \subset H$ is given by $[(x, n), (y, m)] = B(x, y)$. Hence \mathcal{L} on N gives rise to a Heisenberg idempotent and the corresponding metric group is the anisotropic group $(\mathbb{F}_{p^2}, \theta)$. Note that we have $\gamma(B) = B$ and that $\gamma(xy^p) = xy^p$ for all $\gamma \in \Gamma$ and $x, y \in \mathbb{G}_a$ since $xy^p \in \mathbb{G}_a \subset N$ and by construction $\mathbb{G}_a \subset N$ is fixed by the Γ -action.

The group Γ acts on \tilde{N}, N and hence it acts on the groups $\text{Hom}(\mathbb{G}_a, \tilde{N}), \text{Hom}(\mathbb{G}_a, N)$. Let us form the semidirect product $\text{Hom}(\mathbb{G}_a, N) \rtimes \Gamma$. Note that we have an action of this semidirect product on H as follows:

$$\text{Hom}(\mathbb{G}_a, N) \rtimes \Gamma \ni (\phi, \gamma) : (y, m) \mapsto (y, \gamma(m) + \phi(y)).$$

It is easy to check that this is indeed an action and that the induced action on H/N is trivial and the induced action of Γ on N is the original action. Note that the inner action of \mathbb{G}_a on H is also given by the above action of $\text{Hom}(\mathbb{G}_a, N)$ on H via the embedding $\mathbb{G}_a \xrightarrow{x \mapsto B(x, \cdot)} \text{Hom}(\mathbb{G}_a, N)$.

Note that we have the Γ -equivariant map $\mathbb{F}_{p^2} \rightarrow \tilde{N}$ given by $k \mapsto k^p \in \mathbb{G}_a \subset \tilde{N}$. Hence pushing forward the extension $0 \rightarrow \mathbb{F}_{p^2} \rightarrow K_\Gamma \rightarrow \Gamma \rightarrow 0$ along this map we get an extension $0 \rightarrow \tilde{N} \rightarrow M \rightarrow \Gamma \rightarrow 0$. Since \tilde{N} has trivial cohomology, this must be the semidirect product and we must have $M \cong \tilde{N} \rtimes \Gamma$. Note that we have the sequence of Γ -equivariant maps $\mathbb{F}_{p^2} \rightarrow \tilde{N} \subset \text{Hom}(\mathbb{G}_a, \tilde{N}) \rightarrow \text{Hom}(\mathbb{G}_a, N)$ and the composition is $k \mapsto B(k, \cdot)$. This also gives us a homomorphism $M \cong \tilde{N} \rtimes \Gamma \rightarrow \text{Hom}(\mathbb{G}_a, N) \rtimes \Gamma$ and hence an action of M on H . The map $K_\Gamma \rightarrow M$ gives us an action of K_Γ on H . The induced action of $\mathbb{F}_{p^2} \subset \mathbb{G}_a \cong H/N = \text{Inn}(H)$ on H is just the inner action. Hence we get the desired outer action of Γ on H . We see that K_Γ is exactly the central extension of Γ by $K_\mathcal{L} = \mathbb{F}_{p^2}$ we obtain from this outer action.

3.3.5 Conclusion

Let $f \in H^2(\Gamma, \mathbb{F}_{p^2})$ be such that the corresponding obstruction $\beta \in H^4(\Gamma, \overline{\mathbb{Q}}_l^*)$ vanishes. Note that β can be defined as an element of $H^4(\Gamma, \mu_p)$. Hence β must vanish in some $H^4(\Gamma, \mu_{p^n})$. Let $\tilde{N}_n = W_n \Gamma = \text{Maps}(\Gamma, W_n)$, the group algebra of Γ over the ring scheme of Witt vectors of length n . We have $A_n = \mathbb{Z}/p^n \mathbb{Z} \subset W_n \xrightarrow{\Delta} \tilde{N}_n$. Let $N_n = \tilde{N}_n / A_n$. Hence we get a Γ -equivariant central extension $0 \rightarrow A_n \rightarrow \tilde{N}_n \rightarrow N_n \rightarrow 0$, i.e. a Γ -equivariant local system \mathcal{L}_n on N_n (after identifying A_n and μ_{p^n}). Note that we have an embedding $W_n \hookrightarrow N_n$ such that Γ acts trivially on the embedded $W_n \subset N_n$, given by $W_n \cong W_n / A_n \subset \tilde{N}_n / A_n = N_n$ and restricting the above central extension to $W_n \subset N_n$, we get the Artin-Schreier central extension $0 \rightarrow A_n \rightarrow W_n \rightarrow W_n \rightarrow 0$. Hence $\mathcal{L}_n|_{W_n}$ is the Artin-Schreier local system on W_n , which pulls back to the Artin-Schreier local system on \mathbb{G}_a via the embedding $\mathbb{G}_a \hookrightarrow W_n$. Note that as before, we have the skew-symmetric biadditive map $B : \mathbb{G}_a \times \mathbb{G}_a \rightarrow \mathbb{G}_a \subset N \rightarrow N_n$.

Hence we can form the fake Heisenberg group $H_n = \mathbb{G}_a \times N_n$ as before and we can define an outer action of Γ on H_n as before and get the obstruction $\omega \in H^3(\Gamma, N_n)$. Note that since $H^i(\Gamma, \tilde{N}_n) = 0$ by Shapiro's lemma, we have an isomorphism $\delta : H^3(\Gamma, N_n) \cong H^4(\Gamma, A_n)$ coming from the long

exact sequence of group cohomology. Since $\beta = \delta\omega$ vanishes in $H^4(\Gamma, A_n)$, the obstruction ω must also vanish. Hence there exists an extension G_n of Γ by H_n with the Heisenberg admissible pair (N_n, \mathcal{L}_n) such that the corresponding braided Γ -crossed category corresponds to the 2-cocycle f .

Hence using Proposition 3.4, we see that we can realize any braided Γ -crossed category corresponding to $f \in H^2(\Gamma, \mathbb{F}_{p^2})$. Hence by Theorem 3.3 and the classification of such braided Γ -crossed categories, we see that we can realize all modular categories from the class \mathfrak{C}_p^- from unipotent groups and by §3.1, we can pass to a connected unipotent group. Hence the proof of Theorem 1.5 is now complete.

References

- [B] M. Boyarchenko. *Characters of Unipotent Groups over Finite Fields*, Selecta Mathematica, Vol. 16 (2010), No. 4, pp. 857–933, arXiv:0712.2614v4.
- [BD06] M. Boyarchenko and V. Drinfeld. *A motivated introduction to character sheaves on unipotent groups in positive characteristic*, September 2006, arXiv:math/0609769v2.
- [BD08] M. Boyarchenko and V. Drinfeld. *Character Sheaves on Unipotent Groups in Positive Characteristic: Foundations*, October 2008, arXiv:0810.0794v1.
- [Da] S. Datta. *Metric Groups attached to Skew-symmetric Biextensions*, Transformation Groups (2010), arXiv:0809.5082v2.
- [De] T. Deshpande. *Heisenberg Idempotents on Unipotent Groups*, Math. Res. Lett. 17 (2010), no. 3, 415 - 434, arXiv:0907.3344.
- [ENO] P. Etingof, D. Nikshych, V. Ostrik. *Fusion Categories and Homotopy Theory*, September 2009, arXiv:0909.3140v2.
- [DGNO] V. Drinfeld, S. Gelaki, D. Nikshych and V. Ostrik. *On Braided Fusion Categories I*, June 2009, arXiv:0906.0620v1.
- [DGNO2] V. Drinfeld, S. Gelaki, D. Nikshych and V. Ostrik. *On Braided Fusion Categories II*, In preparation.